Pergamon

Int. J. Solids Structures Vol. 32, No. 22, pp. 3327–3340, 1995

0020-7683/95 $9.50 + .00

0020–7683(94)00307–6

UNIFIED EXPLICIT BASIS-FREE EXPRESSIONS FOR TIME RATE AND CONJUGATE STRESS OF AN ARBITRARY HILL'S STRAIN

H. XIAO
Department of Mathematics & Institute of Mathematics, Peking University, Beijing 100871, China

(*Received* 17 *May* 1994; *in revised form* 20 *November* 1994)

Abstract—This paper provides simple and unified explicit basis-free expressions for time rate and conjugate stress of an arbitrary Hill's strain for the first time, which are valid for all cases of the eigenvalues of the right stretch tensors.

1. INTRODUCTION

Let $\mathbf{U}$ be the right stretch tensor and $\{\lambda_i\}$ and $\{\mathbf{N}_i\}$ be the eigenvalues of $\mathbf{U}$ and the subordinate orthonormal eigenvectors, respectively. The following class of strain measures is known as Hill's strains [cf. Hill (1968, 1978)]:

$$\mathbf{E} = \mathbf{E}(\mathbf{U}) = \sum_{i=1}^{d} f(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i, \quad d = 3, \tag{1a,b}$$

where $f(\cdot)$ is a strictly-increasing scalar function satisfying $f(1) = 0$ and $f'(1) = 1$ [henceforth we shall assume that $f(\cdot)$ is at least of C^1]. The following subclass of Hill's strains, indexed by the parameter m, is known as Seth's strains [cf. Seth (1964)]:

$$\mathbf{E}^{(m)} = \frac{1}{m}\sum_{i=1}^{3}(\lambda_i^m - 1)\mathbf{N}_i \otimes \mathbf{N}_i = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}). \tag{2}$$

Hill's strains, even Seth's strains, are broad enough to include almost all the commonly used Lagrangian-type strains such as the nominal strain $\mathbf{E}^{(1)}$, Green's strain $\mathbf{E}^{(2)}$, Almansi's strain $\mathbf{E}^{(-2)}$ and the logarithmic strain $\mathbf{E}^{(0)} = \ln \mathbf{U}$, etc.

On the other hand, by means of the notion of work conjugacy, introduced by Hill (1968) and Macvean (1968), a class of stress measures may be derived in a natural way. Let $\mathbf{E}$ be a Lagrangian-type strain. A symmetric second order tensor $\mathbf{T}$ is the conjugate stress of the strain measure $\mathbf{E}$ if $\mathbf{T}:\dot{\mathbf{E}}$ offers the stress power w per unit reference state volume, i.e.

$$w = \mathrm{III}\boldsymbol{\sigma}:\mathbf{D} = \mathbf{T}:\dot{\mathbf{E}}, \tag{3}$$

where $\boldsymbol{\sigma}$, $\mathbf{D}$ and III are the Cauchy stress, the stretching tensor and the third principal invariant of $\mathbf{U}$, respectively.

The conjugate stresses of $\mathbf{E}^{(1)}$, $\mathbf{E}^{(2)}$ and $\mathbf{E}^{(-2)}$ are well known [cf. Hill (1978) and Guo (1984)]: they are the second Piola–Kirchhoff stress tensor,

$$\mathbf{T}^{(2)} = \mathrm{III}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}, \tag{4}$$

the weighted convected stress tensor,

$$\mathbf{T}^{(-2)} = \mathrm{III}\,\mathbf{F}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{F}, \tag{5}$$

and the Jaumann stress tensor

$$\mathbf{T}^{(1)} = \tfrac{1}{2}(\mathbf{T}^{(2)}\mathbf{U}+\mathbf{U}\mathbf{T}^{(2)}). \tag{6}$$

Here $\mathbf{F}$ is the deformation gradient.

In the theory of finite deformations and in constitutive modelling, the aforementioned strain measures, their time rates and their conjugate stresses are basic [cf. Hill (1968, 1978), Guo and Dubey (1984), etc.]. The problem of finding expressions for these basic quantities has attracted many researchers' attention in past decades, especially in the last decade. By using the principal axis method, Hill (1968, 1978) derived component expressions for $\mathbf{E}(\mathbf{U})$ and some conjugate stresses [cf. Guo and Dubey (1984) and Scheidler (1991a) for a more compact form of the results]. However, component expressions, as given in a principal basis, are valid only in this basis. In field problems (problems in solid mechanics or continuum mechanics are often field problems), they are generally not satisfactory, because at each point they demand a principal basis consisting of three orthonormal eigenvectors of $\mathbf{U}$. Although one can attack this by determining the eigenvalues and the corresponding eigenvectors leading to the orthogonal transformation between the principal and common bases, such procedures are usually tedious. As a result, the need to find explicit basis-free expressions for the aforementioned basic quantities, which avoid the aforementioned tedious procedures, become pressing.

Gurtin and Spear (1983) discussed the relationship between the logarithmic strain rate and the stretching tensor. Guo (1984) first provided basis-free expressions for rates of the stretch tensors [see also Guo *et al.* (1991)]. Carlson and Hoger (1986), Hoger and Carlson (1984b), Hoger (1986), Mehrabadi and Nemat-Nassert (1987), Scheidler (1991b, 1992), Wang and Duan (1991) and Man and Guo (1993) obtained many results for basis-free expressions for various strain measure rates. On the other hand, Hoger (1987) and Lehmann and Liang (1993) offered basis-free expressions for the conjugate stresses of the logarithmic strains ln $\mathbf{U}$ and ln $\mathbf{V}$, respectively. The basis-free expression for the conjugate stress of an arbitrary Seth strain was derived by Guo and Man (1992) [cf. Guo *et al.* (1994)].

Thus far, the conjugate stress of an arbitrary Hill's strain has not yet been available. Moreover, the existing results provide distinct expressions for distinct cases when the eigenvalues of $\mathbf{U}$ are distinct, doubly coalecent and triply coalecent. Unified basis-free expressions for time rates and conjugate stresses of various strain measures, valid for all cases of the eigenvalues of $\mathbf{U}$, are still wanting.

The objective of the paper is to provide simple and unified explicit basis-free expressions for time rate and conjugate stress of an arbitrary Hill's strain. The main procedures are as follows. In Section 2, as a basis for the subsequent sections we find the expression for the twirl tensor $\boldsymbol{\Omega}$ which rotates the Lagrangian triad $\{\mathbf{N}_i\}$ consisting of three orthonormal eigenvectors of $\mathbf{U}$. In Section 3, we offer a simple and unified basis-free expression for time rate of an arbitrary Hill's strain. In Section 4, we derive a simple and unified basis-free expression for conjugate stress of an arbitrary Hill's strain from the notion of work conjugacy. Finally, in Section 5, we discuss some examples for illustration.

2. TWIRL TENSOR

In a deforming body, the Lagrangian triad $\{\mathbf{N}_i\}$ rotates with respect to a fixed orthogonal triad. The angular velocity of $\{\mathbf{N}_i\}$ can be described by a skew-symmetric second order tensor $\boldsymbol{\Omega}$, i.e.

$$\dot{\mathbf{N}}_i = \boldsymbol{\Omega}\mathbf{N}_i. \tag{7}$$

$\boldsymbol{\Omega}$, called the twirl tensor by Hoger (1986) and Guo *et al.* (1992), will be used in the next section. If we can justifiably differentiate the spectral representation

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i \tag{8}$$

with respect to the time t, then we get

$$\begin{aligned}\dot{\mathbf{U}} &= \sum_{i=1}^{3} (\dot{\lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i + \lambda_i (\mathbf{\Omega N}_i) \otimes \mathbf{N}_i + \lambda_i \mathbf{N}_i \otimes (\mathbf{\Omega N}_i)) \\ &= \sum_{i=1}^{3} \dot{\lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i + \mathbf{\Omega U} - \mathbf{U\Omega},\end{aligned}$$

i.e.

$$\mathbf{\Omega U} - \mathbf{U\Omega} = \dot{\mathbf{U}} - \sum_{i=1}^{3} \dot{\lambda}_i \mathbf{N}_i \otimes \mathbf{N}_i. \tag{9}$$

In the preceding equation, $\mathbf{U}$ is supposed to be a given function of t, which is at least of class C^1.

Guo *et al.* (1992) derived a similar equation for the twirl tensor of any second order symmetric tensor describing physical phenomena and offered its distinct basis-free solutions for distinct cases of the eigenvalues of $\mathbf{U}$. Recently, Guo *et al.* (1995) further studied the n-dimensional tensor equation $\mathbf{AX}-\mathbf{XA} = \mathbf{C}$, where $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{X}$ are second order tensors over an n-dimensional Euclidean space and $\mathbf{A}$ is symmetric, and derived a unified basis-free solution, valid for all cases of the eigenvalues of $\mathbf{A}$. In the following, we shall investigate the preceding tensor equation using a new method.

Let $\lambda_1, \ldots, \lambda_m$ be all the distinct eigenvalues of $\mathbf{U}$ and $\mathbf{P}_1, \ldots, \mathbf{P}_m$ be the subordinate eigenprojections. Then eqns (1) and (8) may be recast into

$$\mathbf{U} = \sum_{\sigma=1}^{m} \lambda_\sigma \mathbf{P}_\sigma ; \quad \mathbf{E} = \sum_{\sigma=1}^{m} f(\lambda_\sigma) \mathbf{P}_\sigma, \tag{10a,b}$$

with

$$\mathbf{P}_\sigma \mathbf{P}_\tau = \delta_{\sigma\tau} \mathbf{P}_\tau (\delta_{\sigma\tau} \text{ is the Kronecker delta}) \tag{11}$$

$$\sum_{\sigma=1}^{m} \mathbf{P}_\sigma = \mathbf{I}, \tag{12}$$

where $\mathbf{I}$ is the second order identity tensor. The eigenprojections are expressible in terms of $\mathbf{U}$ and its eigenvalues. In reality [cf. Luehr and Rubin (1990)],

$$\mathbf{P}_\sigma = \prod_{\substack{\tau=1 \\ \tau \neq \sigma}}^{m} \frac{\mathbf{U} - \lambda_\tau \mathbf{I}}{\lambda_\sigma - \lambda_\tau}, \quad m > 1 ; \quad \mathbf{P}_\sigma = \mathbf{I}, \quad m = 1 ; \tag{13}$$

hence,

$$\mathbf{P}_\sigma = \frac{1}{p_\sigma}\sum_{r=0}^{m-1} I^\sigma_{m-1-r}\mathbf{U}^r \tag{14}$$

$$\begin{cases} P_\sigma = \prod_{\substack{\tau=1\\ \tau\neq\sigma}}^{m} (\lambda_\sigma-\lambda_\tau) \neq 0;\, I^\sigma_0 = 1; \quad \text{and for } s = 1,\ldots,m-1, \\ I^\sigma_s = (-1)^s \sum_{1\leq\sigma_1<\cdots<\sigma_s\leq m} \lambda_{\sigma_1}\ldots\lambda_{\sigma_s}(1-\delta_{\sigma\sigma_1})\ldots(1-\delta_{\sigma\sigma_s}). \end{cases} \tag{15}$$

By using eqn (10a), we recast eqn (9) into

$$\mathbf{\Omega U}-\mathbf{U\Omega} = \dot{\mathbf{U}} - \sum_{\sigma=1}^{m} \dot{\lambda}_\sigma \mathbf{P}_\sigma. \tag{16}$$

By eqns (10a) and (11), we infer:

$$\mathbf{P}_\sigma\mathbf{U} = \mathbf{U}\mathbf{P}_\sigma = \lambda_\sigma\mathbf{P}_\sigma, \sigma = 1,\ldots,m. \tag{17}$$

From this and eqn (16) we derive:

$$\mathbf{P}_\tau\left(\dot{\mathbf{U}} - \sum_{\sigma=1}^{m}\dot{\lambda}_\sigma\mathbf{P}_\sigma\right)\mathbf{P}_\tau = \mathbf{P}_\tau(\mathbf{\Omega U}-\mathbf{U\Omega})\mathbf{P}_\tau = \mathbf{O}, \quad \tau = 1,\ldots,m,$$

i.e.

$$\dot{\lambda}_\sigma\mathbf{P}_\sigma = \mathbf{P}_\sigma\dot{\mathbf{U}}\mathbf{P}_\sigma, \quad \sigma = 1,\ldots,m. \tag{18}$$

By means of eqn (1) it can be readily proved that the above m conditions are equivalent to the following:

$$\sum_{\sigma=1}^{m} \mathbf{P}_\sigma\dot{\mathbf{U}}\mathbf{P}_\sigma = \sum_{\sigma=1}^{m} \dot{\lambda}_\sigma\mathbf{P}_\sigma. \tag{19}$$

The latter is the necessary and sufficient condition for the existence of the solution of the tensor equation (16). We have the following result.

Theorem 1. The tensor equation (16) has a solution iff eqn (19) or (18) holds and under this condition the general solution $\mathbf{\Omega}\in skw$ of the tensor equation (16) is of the following form:

$$\mathbf{\Omega} = \sum_{\sigma=1}^{m} \mathbf{P}_\sigma\mathbf{W}\mathbf{P}_\sigma + \sum_{\sigma\neq\tau}\frac{1}{\lambda_\tau-\lambda_\sigma}\mathbf{P}_\sigma\dot{\mathbf{U}}\mathbf{P}_\tau, \quad \forall\,\mathbf{W}\in skw. \tag{20}$$

Here $\Sigma_{\sigma\neq\tau}$ means that the summation indices σ, $\tau = 1,\ldots,m$ and $\sigma\neq\tau$. For $m = 1$, such summation is taken as zero. Here, skw is used to represent the set of all skew-symmetric second order tensors.

Prior to the proof, we make the following convention. *Lin* represents the second order tensor space over a three-dimensional Euclidean space. The sign : indicates double dot or interior product of two tensors. In addition, two juxtaposed tensors mean their compositional product (this convention has been used).

The proof of Theorem 1 is as follows.

Define a bilinear map $\boxtimes : Lin \times Lin \to L(Lin) : (\mathbf{A},\mathbf{B}) \mapsto \mathbf{A}\boxtimes\mathbf{B}$ by

$$(\mathbf{A} \boxtimes \mathbf{B}) : \mathbf{X} = \mathbf{AXB}^{T}, \quad \forall \mathbf{X} \in Lin. \tag{21}$$

Obviously, the fourth-order tensor over R^3, $\mathbf{I} \boxtimes \mathbf{I}$, is the second-order identity tensor over *Lin*. We mention that $L(Lin)$ is the fourth-order tensor space over R^3, i.e. the second-order tensor space over *Lin*.

By using eqns (10a) and (12) we get

$$\mathbf{I} \boxtimes \mathbf{U} - \mathbf{U} \boxtimes \mathbf{I} = \left(\sum_{\sigma=1}^{m} \mathbf{P}_\sigma \right) \boxtimes \left(\sum_{\tau=1}^{m} \lambda_\tau \mathbf{P}_\tau \right) - \left(\sum_{\sigma=1}^{m} \lambda_\sigma \mathbf{P}_\sigma \right) \boxtimes \left(\sum_{\tau=1}^{m} \mathbf{P}_\tau \right)$$

$$= \sum_{\sigma,\tau=1}^{m} (\lambda_\tau - \lambda_\sigma) \mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau = \sum_{\sigma \neq \tau} (\lambda_\tau - \lambda_\sigma) \mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau. \tag{22}$$

From the following facts:

$$(\mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau) : (\mathbf{P}_{\sigma'} \boxtimes \mathbf{P}_{\tau'}) = \begin{cases} \mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau, & \sigma = \sigma', \tau = \tau' \\ \mathbf{O}, & \text{otherwise} \end{cases}$$

$$\sum_{\sigma,\tau=1}^{m} \mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau = \mathbf{I} \boxtimes \mathbf{I},$$

we conclude that the right-hand side of eqn (22) offers the spectral representation of the second order tensor over *Lin*, $\mathbf{I} \boxtimes \mathbf{U} - \mathbf{U} \boxtimes \mathbf{I}$, and hence this tensor is a symmetric second-order tensor. The tensor equation (16) can be written into the following vector equation over *Lin*:

$$(\mathbf{I} \boxtimes \mathbf{U} - \mathbf{U} \boxtimes \mathbf{I}) : \mathbf{\Omega} = \dot{\mathbf{U}} - \sum_{\sigma=1}^{m} \dot{\lambda}_\sigma \mathbf{P}_\sigma (= \mathbf{C}). \tag{23}$$

Hence, the tensor equation (16) has a solution iff $\mathbf{C}$ is contained in the image space of $\bar{\mathbf{U}} = \mathbf{I} \boxtimes \mathbf{U} - \mathbf{U} \boxtimes \mathbf{I}$, i.e. $\mathbf{C} \in \mathrm{Im}(\bar{\mathbf{U}})$. Since the image space $\mathrm{Im}(\bar{\mathbf{U}})$ and the kernel space $\mathrm{Ker}(\bar{\mathbf{U}})$ of the second-order symmetric tensor $\bar{\mathbf{U}}$ is orthogonal and $Lin = \mathrm{Im}(\bar{\mathbf{U}}) \oplus \mathrm{Ker}(\bar{\mathbf{U}})$, as well as the orthogonal projection of $\mathrm{Ker}(\bar{\mathbf{U}})$, is $\Sigma_{\sigma=1}^{m} \mathbf{P}_\sigma \boxtimes \mathbf{P}_\sigma$, we infer that $\mathbf{C} \in \mathrm{Im}(\bar{\mathbf{U}})$ iff $\mathbf{C}$ is perpendicular to the kernel space $\mathrm{Ker}(\bar{\mathbf{U}})$, i.e.

$$\sum_{\sigma=1}^{m} (\mathbf{P}_\sigma \boxtimes \mathbf{P}_\sigma) : \mathbf{C} = \mathbf{0},$$

i.e. eqn (19) holds. Moreover, since the restriction $\bar{\mathbf{U}}|_{\mathrm{Im}(\bar{\mathbf{U}})} : \mathrm{Im}(\bar{\mathbf{U}}) \to \mathrm{Im}(\bar{\mathbf{U}})$ is a nonsingular linear transformation over $\mathrm{Im}(\bar{\mathbf{U}})$ and by means of the spectral representation (22) its inverse is readily available, hence eqn (16), i.e. eqn (23), has a unique solution contained in the image space $\mathrm{Im}(\bar{\mathbf{U}})$, given by

$$\mathbf{\Omega}^{\mathrm{Im}} = \left(\sum_{\sigma \neq \tau} (\lambda_\tau - \lambda_\sigma)^{-1} \mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau \right) : \mathbf{C} = \sum_{\sigma \neq \tau} (\lambda_\tau - \lambda_\sigma)^{-1} \mathbf{P}_\sigma \mathbf{C} \mathbf{P}_\tau = -(\mathbf{\Omega}^{\mathrm{Im}})^{T}. \tag{24}$$

Thus, the general solution of eqn (16), i.e. eqn (23), is as follows:

$$\mathbf{\Omega} = \mathbf{\Omega}^{\mathrm{Ker}} + \mathbf{\Omega}^{\mathrm{Im}} \text{ for each } \mathbf{\Omega}^{\mathrm{Ker}} \in \mathrm{Ker}(\bar{\mathbf{U}}) \cap skw = \left\{ \sum_{\sigma=1}^{m} \mathbf{P}_\sigma \mathbf{W} \mathbf{P}_\sigma \,\middle|\, \mathbf{W} \in skw \right\},$$

so eqn (20) holds.

Remark. Equation (20) offers the general solution of the tensor equation (16) in explicit form, valid for all cases of the eigenvalues of **U**. This result, as it will be seen, plays a key role in the succeeding discussion. As far as we know, it is given for the first time.

Remark. If **U** has three distinct eigenvalues, then for each $\mathbf{W} \in skw$,

$$\sum_{\sigma=1}^{m} \mathbf{P}_\sigma \mathbf{W} \mathbf{P}_\sigma = \sum_{i=1}^{3} (\mathbf{N}_i \mathbf{W} \mathbf{N}_i) \mathbf{N}_i \otimes \mathbf{N}_i = \mathbf{0}. \tag{25}$$

Thus, the solution of eqn (16) is unique and hence eqn (20) offers the required twirl tensor. However, if **U** has repeated eigenvalues, eqn (25) no longer holds and hence eqn (16) has an infinite number of solutions. Guo *et al.* (1992) have pointed out that the skew-symmetric tensor $\mathbf{\Omega}^{\mathrm{Im}}$ may be justifiably taken as the twirl tensor for the case when **U** has repeated eigenvalues. In our succeeding analysis, the multivaluedness of the solution of eqn (16) is of no consequence and hence such further considerations are not essential, so we do not bother to list further related results. Moreover, we obtain eqn (9) by differentiating eqn (8) with respect to time and by using eqn (7), the defining equation for $\mathbf{\Omega}$. There remains the question of whether and under what conditions we may justifiably proceed as indicated. This problem has been solved by Guo *et al.* (1992).

3. TIME RATE OF AN ARBITRARY HILL'S STRAIN

Carlson and Hoger (1986) and Man and Guo (1993) provided distinct basis-free expressions for the time rate of an arbitrary Hill's strain for distinct cases of the eigenvalues of **U** [cf. Wang and Duan (1991) and Scheidler (1991b, 1992)]. In this section, we provide new and unified basis-free expressions, which are valid for all cases of the eigenvalues of **U** and from which the conjugate stress of an arbitrary Hill's strain can be derived immediately.

By differentiating eqn (1) and by using eqns (7) and (10a), we obtain

$$\dot{\mathbf{E}} = \sum_{\sigma=1}^{m} f'(\lambda_\sigma)\dot{\lambda}_\sigma \mathbf{P}_\sigma + \mathbf{\Omega}\mathbf{E} - \mathbf{E}\mathbf{\Omega}. \tag{26}$$

Substituting eqn (18) into the above equation, we further get

$$\dot{\mathbf{E}} = \sum_{\sigma=1}^{m} f'(\lambda_\sigma)\mathbf{P}_\sigma \dot{\mathbf{U}} \mathbf{P}_\sigma + \mathbf{\Omega}\mathbf{E} - \mathbf{E}\mathbf{\Omega}. \tag{27}$$

By eqns (10b) and (11) we infer:

$$\mathbf{\Omega}^{\mathrm{Ker}}\mathbf{E} - \mathbf{E}\mathbf{\Omega}^{\mathrm{Ker}} = \left(\sum_{\sigma=1}^{m} \mathbf{P}_\sigma \mathbf{W} \mathbf{P}_\sigma\right)\mathbf{E} - \mathbf{E}\left(\sum_{\sigma=1}^{m} \mathbf{P}_\sigma \mathbf{W} \mathbf{P}_\sigma\right) = \mathbf{0}, \quad \forall \mathbf{W} \in skw. \tag{28}$$

Hence, by substituting eqn (20) into (27) and then using eqns (28) and (11), we get

$$\begin{aligned}
\dot{\mathbf{E}} = {} & \sum_{\sigma=1}^{m} f'(\lambda_\sigma)\mathbf{P}_\sigma \dot{\mathbf{U}} \mathbf{P}_\sigma + \left(\sum_{\sigma \neq \tau} (\lambda_\tau - \lambda_\sigma)^{-1} \mathbf{P}_\sigma \dot{\mathbf{U}} \mathbf{P}_\tau\right)\left(\sum_{\theta=1}^{m} f(\lambda_\theta)\mathbf{P}_\theta\right) \\
& - \left(\sum_{\theta=1}^{m} f(\lambda_\theta)\mathbf{P}_\theta\right)\left(\sum_{\sigma \neq \tau} (\lambda_\tau - \lambda_\sigma)^{-1} \mathbf{P}_\sigma \dot{\mathbf{U}} \mathbf{P}_\tau\right) \\
= {} & \sum_{\sigma=1}^{m} f'(\lambda_\sigma)\mathbf{P}_\sigma \dot{\mathbf{U}} \mathbf{P}_\sigma + \sum_{\sigma \neq \tau} \frac{f(\lambda_\sigma) - f(\lambda_\tau)}{\lambda_\sigma - \lambda_\tau} \mathbf{P}_\sigma \dot{\mathbf{U}} \mathbf{P}_\tau.
\end{aligned} \tag{29}$$

Denoting

$$f_{\sigma\tau} = \frac{f(\lambda_\sigma)-f(\lambda_\tau)}{\lambda_\sigma-\lambda_\tau}, \quad \text{for } \sigma,\tau = 1,\ldots,m, \tag{30}$$

where for $\sigma = \tau = 1,\ldots,m$, the limiting process $\lambda_\sigma \to \lambda_\tau$ is meant, we may recast the preceding expression into

$$\dot{\mathbf{E}} = \sum_{\sigma,\tau=1}^{m} f_{\sigma\tau}\mathbf{P}_\sigma\dot{\mathbf{U}}\mathbf{P}_\tau = \mathbf{L}(\mathbf{U}):\dot{\mathbf{U}} \tag{31a,b}$$

$$\mathbf{L}(\mathbf{U}) = \sum_{\sigma,\tau=1}^{m} f_{\sigma\tau}\mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau. \tag{32}$$

Finally, substituting eqn (13) into (31a), we obtain the following result.

Theorem 2. The time rate of an arbitrary Hill's strain possesses the following basis-free expression:

$$\dot{\mathbf{E}} = \sum_{r,s=0}^{m-1} \varepsilon_{rs}\mathbf{U}^r\dot{\mathbf{U}}\mathbf{U}^s \tag{33}$$

$$\varepsilon_{rs} = \sum_{\sigma,\tau=1}^{m} f_{\sigma\tau}p_\sigma^{-1}p_\tau^{-1}I^\sigma_{m-1-r}I^\tau_{m-1-s} = \varepsilon_{sr}, r,s = 0,\ldots,m-1. \tag{34}$$

Specifically, we have the following.

(i) $\mathbf{U}$ has three distinct eigenvalues, i.e. $m = 3$; $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

$$\dot{\mathbf{E}} = \varepsilon_{00}\dot{\mathbf{U}}+\varepsilon_{11}\mathbf{U}\dot{\mathbf{U}}\mathbf{U}+\varepsilon_{22}\mathbf{U}^2\dot{\mathbf{U}}\mathbf{U}^2+\varepsilon_{01}(\mathbf{U}\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}) \\ +\varepsilon_{02}(\mathbf{U}^2\dot{\mathbf{U}}+\dot{\mathbf{U}}\mathbf{U}^2)+\varepsilon_{12}(\mathbf{U}\dot{\mathbf{U}}\mathbf{U}^2+\mathbf{U}^2\dot{\mathbf{U}}\mathbf{U}) \tag{35}$$

$$\varepsilon_{00} = \frac{1}{\Delta^2}\sum_{12,23,31}\lambda_\sigma^2\lambda_\tau^2(\lambda_\sigma-\lambda_\tau)^2(f'(\lambda_\sigma)f'(\lambda_\tau))^{-1}\chi_f \\ +\frac{2\mathrm{III}^2}{\Delta}\sum_{12,23,31}(\lambda_\sigma\lambda_\tau)^{-1}(\lambda_\sigma-\lambda_\tau)^{-2}(f(\lambda_\sigma)-f(\lambda_\tau)) \tag{36}$$

$$\varepsilon_{11} = \frac{1}{\Delta^2}\sum_{12,23,31}(\lambda_\sigma^2-\lambda_\tau^2)^2(f'(\lambda_\sigma)f'(\lambda_\tau))^{-1}\chi_f \\ +\frac{2\Gamma}{\Delta}\sum_{12,23,31}(\lambda_\sigma+\lambda_\tau)^{-1}(\lambda_\sigma-\lambda_\tau)^{-2}(f(\lambda_\sigma)-f(\lambda_\tau)) \tag{37}$$

$$\varepsilon_{22} = \frac{1}{\Delta^2}\sum_{12,23,31}(\lambda_\sigma-\lambda_\tau)^2(f'(\lambda_\sigma)f'(\lambda_\tau))^{-1}\chi_f+\frac{2}{\Delta}\sum_{12,23,31}(\lambda_\sigma-\lambda_\tau)^{-2}(f(\lambda_\sigma)-f(\lambda_\tau)) \tag{38}$$

$$\varepsilon_{01} = -\frac{1}{\Delta^2}\sum_{12,23,31}\lambda_\sigma\lambda_\tau(\lambda_\sigma+\lambda_\tau)(\lambda_\sigma-\lambda_\tau)^2(f'(\lambda_\sigma)f'(\lambda_\tau))^{-1}\chi_f \\ -\frac{\mathrm{III}}{\Delta}\sum_{12,23,31}(2+\mathrm{III}\lambda_\sigma^{-2}\lambda_\tau^{-2}(\lambda_\sigma+\lambda_\tau))(\lambda_\sigma-\lambda_\tau)^{-2}(f(\lambda_\sigma)-f(\lambda_\tau)) \tag{39}$$

$$\varepsilon_{02} = \frac{1}{\Delta^2} \sum_{12,23,31} \lambda_\sigma \lambda_\tau (\lambda_\sigma - \lambda_\tau)^2 (f'(\lambda_\sigma) f'(\lambda_\tau))^{-1} \chi_f + \frac{\text{III}}{\Delta} \sum_{12,23,31} \lambda_\sigma^{-1} \lambda_\tau^{-1} (\lambda_\sigma + \lambda_\tau)(\lambda_\sigma - \lambda_\tau)^{-2} (f(\lambda_\sigma) - f(\lambda_\tau)) \quad (40)$$

$$\varepsilon_{12} = -\frac{1}{\Delta^2} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau)(\lambda_\sigma - \lambda_\tau)^2 (f'(\lambda_\sigma) f'(\lambda_\tau))^{-1} \chi_f - \frac{1}{\Delta} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau + 2\text{III} \lambda_\sigma^{-1} \lambda_\tau^{-1})(\lambda_\sigma - \lambda_\tau)^{-2} (f(\lambda_\sigma) - f(\lambda_\tau)), \quad (41)$$

where $\sum_{12,23,31} J_{\sigma\tau} = J_{12} + J_{23} + J_{31}$ and

$$\begin{cases} \Delta = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \\ \chi_f = f'(\lambda_1) f'(\lambda_2) f'(\lambda_3) \\ \Gamma = (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) = \text{I II} - \text{III} \end{cases} \quad (42)$$

and I, II and III are the three principal invariants of $\mathbf{U}$, i.e.

$$\begin{cases} \text{I} = \lambda_1 + \lambda_2 + \lambda_3 \\ \text{II} = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 \\ \text{III} = \lambda_1 \lambda_2 \lambda_3. \end{cases} \quad (43)$$

(ii) $\mathbf{U}$ has only two distinct eigenvalues, i.e. $m = 2$; $\lambda_1 \neq \lambda_2$.

$$\dot{\mathbf{E}} = \varepsilon_{00} \dot{\mathbf{U}} + \varepsilon_{11} \mathbf{U} \dot{\mathbf{U}} \mathbf{U} + \varepsilon_{01} (\mathbf{U} \dot{\mathbf{U}} + \dot{\mathbf{U}} \mathbf{U}) \quad (44)$$

$$\varepsilon_{00} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\lambda_1^2 f'(\lambda_2) + \lambda_2^2 f'(\lambda_1) - \frac{2\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} (f(\lambda_1) - f(\lambda_2)) \right) \quad (45)$$

$$\varepsilon_{11} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(f'(\lambda_1) + f'(\lambda_2) - 2 \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2} \right) \quad (46)$$

$$\varepsilon_{01} = \frac{-1}{(\lambda_1 - \lambda_2)^2} \left(\lambda_1 f'(\lambda_2) + \lambda_2 f'(\lambda_1) - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} (f(\lambda_1) - f(\lambda_2)) \right). \quad (47)$$

(iii) $\mathbf{U} = \lambda \mathbf{I}$, i.e. $m = 1$.

$$\dot{\mathbf{E}} = \varepsilon_{00} \dot{\mathbf{U}} = f'(\lambda) \dot{\mathbf{U}}. \quad (48)$$

4. CONJUGATE STRESS OF AN ARBITRARY HILL'S STRAIN

Let $\mathbf{T}$ be the conjugate stress of the Hill's strain $\mathbf{E}$. According to the notion of work conjugacy [cf. eqn (3)], we have

$$\mathbf{T} : \dot{\mathbf{E}} = \mathbf{T}^{(1)} : \dot{\mathbf{U}}. \quad (49)$$

Here, the Jaumann stress $\mathbf{T}^{(1)}$, given by eqns (4) and (6), is conjugate to the nominal strain $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$.

Substituting eqn (31) into (49), we get

$$(\mathbf{L}(\mathbf{U}):\mathbf{T}):\dot{\mathbf{U}} = \mathbf{T}^{(1)}:\dot{\mathbf{U}}. \tag{50}$$

By the arbitrariness of $\dot{\mathbf{U}}$, we infer:

$$\mathbf{L}(\mathbf{U}):\mathbf{T} = \mathbf{T}^{(1)}. \tag{51}$$

Since $f(\cdot)$ is strictly increasing, i.e. $f'(\cdot) > 0$, we infer

$$f_{\sigma\tau} \neq 0, \quad \sigma, \tau = 1, \ldots, m. \tag{52}$$

From this we know that the tensor

$$\mathbf{L}(\mathbf{U}) = \sum_{\sigma,\tau=1}^{m} f_{\sigma\tau}\mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau$$

is a nonsingular symmetric second-order tensor over *Lin*. The crucial point is that the right-hand side of the above formula is just the spectral representation of $\mathbf{L}(\mathbf{U})$ (cf. the proof for Theorem 1). From this fact and eqn (51), we derive the conjugate stress of $\mathbf{E}$ immediately:

$$\mathbf{T} = \mathbf{L}(\mathbf{U})^{-1}:\mathbf{T}^{(1)} = \left(\sum_{\sigma,\tau=1}^{m} f_{\sigma\tau}^{-1}\mathbf{P}_\sigma \boxtimes \mathbf{P}_\tau\right):\mathbf{T}^{(1)}$$

$$= \sum_{\sigma,\tau=1}^{m} f_{\sigma\tau}^{-1}\mathbf{P}_\sigma\mathbf{T}^{(1)}\mathbf{P}_\tau. \tag{53}$$

Substituting eqn (13) into the above expression we get the following result.

Theorem 3. The conjugate stress of an arbitrary Hill's strain possesses the following basis-free expression:

$$\mathbf{T} = \sum_{r,s=0}^{m-1} \rho_{rs}\mathbf{U}^r\mathbf{T}^{(1)}\mathbf{U}^s \tag{54}$$

$$\rho_{rs} = \sum_{\sigma,\tau=1}^{m} f_{\sigma\tau}^{-1}p_\sigma^{-1}p_\tau^{-1}I^\sigma_{m-1-r}I^\tau_{m-1-s} = \rho_{sr}. \tag{55}$$

It can be seen that the coefficients ρ_{rs} can be obtained merely by replacing the coefficients $f_{\sigma\tau}$ on the right-hand side of eqn (34) with $f_{\sigma\tau}^{-1}$. Specifically, we have the following.

(i) $\mathbf{U}$ has three distinct eigenvalues, i.e. $m = 3$; $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

$$\mathbf{T} = \rho_{00}\mathbf{T}^{(1)} + \rho_{11}\mathbf{U}\mathbf{T}^{(1)}\mathbf{U} + \rho_{22}\mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}^2 + \rho_{01}(\mathbf{U}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U})$$
$$+ \rho_{02}(\mathbf{U}^2\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}^2) + \rho_{12}(\mathbf{U}\mathbf{T}^{(1)}\mathbf{U}^2 + \mathbf{U}^2\mathbf{T}^{(1)}\mathbf{U}) \tag{56}$$

$$\rho_{00} = \frac{1}{\Delta^2}\sum_{12,23,31} \lambda_\sigma^2\lambda_\tau^2(\lambda_\sigma - \lambda_\tau)^2 f'(\lambda_\sigma)f'(\lambda_\tau)\chi_f^{-1} + \frac{2\mathrm{III}^2}{\Delta}\sum_{12,23,31}(\lambda_\sigma\lambda_\tau)^{-1}(f(\lambda_\sigma) - f(\lambda_\tau))^{-1} \tag{57}$$

$$\rho_{11} = \frac{1}{\Delta^2}\sum_{12,23,31}(\lambda_\sigma^2 - \lambda_\tau^2)^2 f'(\lambda_\sigma)f'(\lambda_\tau)\chi_f^{-1} + \frac{2\Gamma}{\Delta}\sum_{12,23,31}(\lambda_\sigma + \lambda_\tau)^{-1}(f(\lambda_\sigma) - f(\lambda_\tau))^{-1} \tag{58}$$

$$\rho_{22} = \frac{1}{\Delta^2} \sum_{12,23,31} (\lambda_\sigma - \lambda_\tau)^2 f'(\lambda_\sigma) f'(\lambda_\tau) \chi_f^{-1} + \frac{2}{\Delta} \sum_{12,23,31} (f(\lambda_\sigma) - f(\lambda_\tau))^{\ 1} \tag{59}$$

$$\rho_{01} = -\frac{1}{\Delta^2} \sum_{12,23,31} \lambda_\sigma \lambda_\tau (\lambda_\sigma + \lambda_\tau)(\lambda_\sigma - \lambda_\tau)^2 f'(\lambda_\sigma) f'(\lambda_\tau) \chi_f^{-1}$$
$$- \frac{\mathrm{III}}{\Delta} \sum_{12,23,31} (2 + \mathrm{III}\lambda_\sigma^{-2} \lambda^{-2} (\lambda_\sigma + \lambda_\tau))(f(\lambda_\sigma) - f(\lambda_\tau))^{-1} \tag{60}$$

$$\rho_{02} = \frac{1}{\Delta^2} \sum_{12,23,31} \lambda_\sigma \lambda_\tau (\lambda_\sigma - \lambda_\tau)^2 f'(\lambda_\sigma) f'(\lambda_\tau) \chi_f^{-1}$$
$$+ \frac{\mathrm{III}}{\Delta} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau) \lambda_\sigma^{-1} \lambda_\tau^{-1} (f(\lambda_\sigma) - f(\lambda_\tau))^{-1} \tag{61}$$

$$\rho_{12} = -\frac{1}{\Delta^2} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau) f'(\lambda_\sigma) f'(\lambda_\tau) \chi_f^{-1}$$
$$- \frac{1}{\Delta} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau + 2\mathrm{III}\lambda_\sigma^{-1} \lambda_\tau^{-1})(f(\lambda_\sigma) - f(\lambda_\tau))^{-1}, \tag{62}$$

where χ_f, Δ and Γ, I, II and III are given by eqns (42) and (43).
(ii) **U** has only two distinct eigenvalues, i.e. $m = 2$; $\lambda_1 \neq \lambda_2$.

$$\mathbf{T} = \rho_{00} \mathbf{T}^{(1)} + \rho_{11} \mathbf{U}\mathbf{T}^{(1)}\mathbf{U} + \rho_{01} (\mathbf{U}\mathbf{T}^{(1)} + \mathbf{T}^{(1)}\mathbf{U}) \tag{63}$$

$$\rho_{00} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\frac{\lambda_1^2 f'(\lambda_1) + \lambda_2^2 f'(\lambda_2)}{f'(\lambda_1) \cdot f'(\lambda_2)} - \frac{2\lambda_1 \lambda_2 (\lambda_1 - \lambda_2)}{f(\lambda_1) - f(\lambda_2)} \right) \tag{64}$$

$$\rho_{11} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\frac{f'(\lambda_1) + f'(\lambda_2)}{f'(\lambda_1) \cdot f'(\lambda_2)} - \frac{2(\lambda_1 - \lambda_2)}{f(\lambda_1) - f(\lambda_2)} \right) \tag{65}$$

$$\rho_{01} = \frac{-1}{(\lambda_1 - \lambda_2)^2} \left(\frac{\lambda_1 f'(\lambda_1) + \lambda_2 f'(\lambda_2)}{f'(\lambda_1) \cdot f'(\lambda_2)} - (\lambda_1^2 - \lambda_2^2)(f(\lambda_1) - f(\lambda_2))^{-1} \right). \tag{66}$$

(iii) $\mathbf{U} = \mathbf{I}$, i.e. $m = 1$.

$$\mathbf{T} = \rho_{00} \mathbf{T}^{(1)} = \frac{1}{f'(\lambda)} \mathbf{T}^{(1)}. \tag{67}$$

5. EXAMPLES

As an example, we derive the basis-free expressions for the time rate and the conjugate stress of the logarithmic strain $\mathbf{E}^{(0)} = \ln \mathbf{U}$ by using the results given in the previous sections. We have

$$f(\lambda) = \ln \lambda; \quad f'(\lambda) = \lambda^{-1}. \tag{68}$$

Substituting these into eqns (36)–(41) and (45)–(48), we obtain:

(i) **U** has three distinct eigenvalues.

$$\varepsilon_{00} = \frac{1}{\mathrm{III}\Delta^2} \sum_{12,23,31} \lambda_\sigma^3 \lambda_\tau^3 (\lambda_\sigma - \lambda_\tau)^2 + \frac{2\mathrm{III}^2}{\Delta} \sum_{12,23,31} (\lambda_\sigma \lambda_\tau)^{-1} (\lambda_\sigma - \lambda_\tau)^{-2} \ln \frac{\lambda_\sigma}{\lambda_\tau} \tag{69}$$

$$\varepsilon_{11} = \frac{1}{\mathrm{III}\Delta^2} \sum_{12,23,31} \lambda_\sigma \lambda_\tau (\lambda_\sigma^2 - \lambda_\tau^2)^2 + \frac{2\Gamma}{\Delta} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau)^{-1} (\lambda_\sigma - \lambda_\tau)^{-2} \ln \frac{\lambda_\sigma}{\lambda_\tau} \tag{70}$$

$$\varepsilon_{22} = \frac{1}{\mathrm{III}\Delta^2} \sum_{12,23,31} \lambda_\sigma \lambda_\tau (\lambda_\sigma - \lambda_\tau)^2 + \frac{2}{\Delta} \sum_{12,23,31} (\lambda_\sigma - \lambda_\tau)^{-2} \ln \frac{\lambda_\sigma}{\lambda_\tau} \tag{71}$$

$$\varepsilon_{01} = \frac{-1}{\mathrm{III}\Delta^2} \sum_{12,23,31} \lambda_\sigma^2 \lambda_\tau^2 (\lambda_\sigma + \lambda_\tau)(\lambda_\sigma - \lambda_\tau)^2 - \frac{\mathrm{III}}{\Delta} \sum_{12,23,31} (2 + \mathrm{III}\lambda_\sigma^{-2} \lambda_\tau^{-2} (\lambda_\sigma + \lambda_\tau))(\lambda_\sigma - \lambda_\tau)^{-2} \ln \frac{\lambda_\sigma}{\lambda_\tau} \tag{72}$$

$$\varepsilon_{02} = \frac{1}{\mathrm{III}\Delta^2} \sum_{12,23,31} \lambda_\sigma^2 \lambda_\tau^2 (\lambda_\sigma - \lambda_\tau)^2 + \frac{\mathrm{III}}{\Delta} \sum_{12,23,31} \lambda_\sigma^{-1} \lambda_\tau^{-1} (\lambda_\sigma + \lambda_\tau)(\lambda_\sigma - \lambda_\tau)^{-2} \ln \frac{\lambda_\sigma}{\lambda_\tau} \tag{73}$$

$$\varepsilon_{12} = \frac{1}{\mathrm{III}\Delta^2} \sum_{12,23,31} \lambda_\sigma \lambda_\tau (\lambda_\sigma + \lambda_\tau)(\lambda_\sigma - \lambda_\tau)^2 - \frac{1}{\Delta} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau + 2\mathrm{III}\lambda_\sigma^{-1} \lambda_\tau^{-1})(\lambda_\sigma - \lambda_\tau)^{-2} \ln \frac{\lambda_\sigma}{\lambda_\tau}. \tag{74}$$

Equations (69)–(74) and (35) offer the time rate of ln **U** when **U** has three eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

(ii) **U** has only two distinct eigenvalues.

$$\varepsilon_{00} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\lambda_1^2 \lambda_2^{-1} + \lambda_2^2 \lambda_1^{-1} - \frac{2\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \ln \frac{\lambda_1}{\lambda_2} \right) \tag{75}$$

$$\varepsilon_{11} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\lambda_1^{-1} + \lambda_2^{-1} - 2(\lambda_1 - \lambda_2)^{-1} \ln \frac{\lambda_1}{\lambda_2} \right) \tag{76}$$

$$\varepsilon_{01} = \frac{-1}{(\lambda_1 - \lambda_2)^2} \left(\lambda_1 \lambda_2^{-1} + \lambda_2 \lambda_1^{-1} - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \ln \frac{\lambda_1}{\lambda_2} \right). \tag{77}$$

Equations (75)–(77) and (44) offer the time rate of ln **U** when **U** has only two distinct eigenvalues.

(iii) $\mathbf{U} = \lambda \mathbf{I}$.

$$\dot{\mathbf{E}} = \lambda^{-1} \dot{\mathbf{U}}. \tag{78}$$

Substituting eqn (68) into eqns (57)–(62) and (64)–(67), we obtain the following.

(i) **U** has three distinct eigenvalues.

$$\rho_{00} = \frac{\mathrm{III}}{\Delta^2} \sum_{12,23,31} \lambda_\sigma \lambda_\tau (\lambda_\sigma - \lambda_\tau)^2 + \frac{2\mathrm{III}^2}{\Delta} \sum_{12,23,31} \left(\lambda_\sigma \lambda_\tau \ln \frac{\lambda_\sigma}{\lambda_\tau} \right)^{-1} \tag{79}$$

$$\rho_{11} = \frac{\mathrm{III}}{\Delta^2} \sum_{12,23,31} \lambda_\sigma^{-1} \lambda_\tau^{-1} (\lambda_\sigma^2 - \lambda_\tau^2)^2 + \frac{2\Gamma}{\Delta} \sum_{12,23,31} \left((\lambda_\sigma + \lambda_\tau) \ln \frac{\lambda_\sigma}{\lambda_\tau} \right)^{-1} \tag{80}$$

$$\rho_{22} = \frac{\mathrm{III}}{\Delta^2} \sum_{12,23,31} \lambda_\sigma^{-1} \lambda_\tau^{-1} (\lambda_\sigma - \lambda_\tau)^2 + \frac{2}{\Delta} \sum_{12,23,31} \left(\ln \frac{\lambda_\sigma}{\lambda_\tau} \right)^{-1} \tag{81}$$

$$\rho_{01} = -\frac{\mathrm{III}}{\Delta^2} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau)(\lambda_\sigma - \lambda_\tau)^2 - \frac{\mathrm{III}}{\Delta} \sum_{12,23,31} (2 + \mathrm{III} \lambda_\sigma^{-2} \lambda_\tau^{-2} (\lambda_\sigma + \lambda_\tau)) \left(\ln \frac{\lambda_\sigma}{\lambda_\tau} \right)^{-1} \tag{82}$$

$$\rho_{02} = \frac{\mathrm{III}}{\Delta^2} \sum_{12,23,31} (\lambda_\sigma - \lambda_\tau)^2 + \frac{\mathrm{III}}{\Delta} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau) \left(\lambda_\sigma \lambda_\tau \ln \frac{\lambda_\sigma}{\lambda_\tau} \right)^{-1} \tag{83}$$

$$\rho_{12} = -\frac{\mathrm{III}}{\Delta^2} \sum_{12,23,31} \lambda_\sigma^{-1} \lambda_\tau^{-1} (\lambda_\sigma + \lambda_\tau) - \frac{1}{\Delta} \sum_{12,23,31} (\lambda_\sigma + \lambda_\tau + 2\mathrm{III} \lambda_\sigma^{-1} \lambda_\tau^{-1}) \left(\ln \frac{\lambda_\sigma}{\lambda_\tau} \right)^{-1}. \tag{84}$$

Equations (79)–(84) and (56) offer the conjugate stress of $\ln \mathbf{U}$ when $\mathbf{U}$ has three distinct eigenvalues.

(ii) $\mathbf{U}$ has only two distinct eigenvalues.

$$\rho_{00} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2 - 2\lambda_1 \lambda_2 (\lambda_1 - \lambda_2) \left(\ln \frac{\lambda_1}{\lambda_2} \right)^{-1} \right) \tag{85}$$

$$\rho_{11} = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\lambda_1 + \lambda_2 - 2(\lambda_1 - \lambda_2) \left(\ln \frac{\lambda_1}{\lambda_2} \right)^{-1} \right) \tag{86}$$

$$\rho_{01} = \frac{-1}{(\lambda_1 - \lambda_2)^2} \left(2\lambda_1 \lambda_2 - (\lambda_1^2 - \lambda_2^2) \left(\ln \frac{\lambda_1}{\lambda_2} \right)^{-1} \right). \tag{87}$$

Equations (85)–(87) and (63) offer the conjugate stress of $\ln \mathbf{U}$ when $\mathbf{U}$ has only two eigenvalues $\lambda_1 \neq \lambda_2$.

(iii) $\mathbf{U} = \lambda \mathbf{I}$.

$$\mathbf{T} = \lambda \mathbf{T}^{(1)}. \tag{88}$$

With considerable labor, one can obtain Hoger's (1986) and Man and Guo's (1993) results for the time rate of $\ln \mathbf{U}$ and Hoger's (1987) result for the conjugate stress of $\ln \mathbf{U}$ from the above results, and vice versa.

6. CONCLUDING REMARKS

In the previous sections, basis-free expressions for time rate and conjugate stress of an arbitrary Hill's strain are derived by a simple method. The expressions (33), (34) and (54), (55) are valid for all cases of the eigenvalues of $\mathbf{U}$. By comparing our results for the time rate of Hill's strains with the corresponding results presented by Carlson and Hoger (1986) and Man and Guo (1993), one can see that the former is simpler than the latter. On the other hand, our result for the conjugate stress of Hill's strains is the first one.

To ensure the validity of our results, some condition should be imposed on the function $f(\cdot)$. This problem was investigated by Carlson and Hoger (1986) and Man and Guo (1993).

Let $\mathbf{U}$ be a symmetric second-order tensor over an n-dimensional Euclidean space. Then for $d = n$, eqns (1a, b) offer a general form of isotropic tensor-valued function of a symmetric tensor $\mathbf{U}$. Carlson and Hoger (1986) were the first to obtain the derivative of such a function. They presented explicit results for n distinct eigenvalues and remarked that for any choice of the dimension n and any particular type of coalescence, the corresponding result could be derived by means of the continuity. Moreover, in a remark they conjectured that, "in general, the formula for $D\mathbf{F}(\mathbf{X})[\mathbf{T}]$ depends only on the number of distinct eigenvalues of $\mathbf{X}$ and is independent of the dimension of the underlying space" [see Carlson and Hoger (1986, p. 421)]. It can be easily understood that eqns (33) and (34) hold for the n-dimensional case and hence that they provide the derivative of the tensor-valued function $\mathbf{E}(\mathbf{U})$ that is valid for all cases of the eigenvalues of $\mathbf{U}$. From eqns (33) and (34) it can be seen that the aforementioned Carlson and Hoger conjecture is true.

The basis-free expressions (33) and (54) for the time rate and the conjugate stress of the Hill's strain $\mathbf{E}(\mathbf{U})$ contain only the integral powers of the right stretch tensor $\mathbf{U}$ and depend linearly on the rate $\dot{\mathbf{U}}$ and the Jaumann stress $\mathbf{T}^{(1)}$, respectively; moreover, their coefficients ε_{rs} and ρ_{rs} [cf. eqns (15), (29), (34) and (55)] depend on the distinct eigenvalues of $\mathbf{U}$ only. To apply these expressions, determination of $\mathbf{U}$ and its distinct eigenvalues is required. As we know, $\mathbf{U}$ is related to the right Cauchy–Green tensor $\mathbf{C}$ by

$$\mathbf{U}^2 = \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}. \tag{89}$$

Presuming that the deformation gradient $\mathbf{F}$ is known, from the above we see that $\mathbf{C}$ is easy to calculate, but the calculation of $\mathbf{U}$ is considerably more difficult, since it is the square root of $\mathbf{C}$. This difficulty can be circumvented by means of the works of Hoger and Carlson (1984a) and Sawyers (1986) [cf. Marsden and Hughes (1983, p. 55) and Ting (1985)]. These works enable us to calculate $\mathbf{U}$, $\mathbf{U}^{-1}$ and the rotation $\mathbf{R}$ etc. in terms of the integral powers of $\mathbf{C}$ and the principal invariants of $\mathbf{C}$. In the following we indicate that the coefficients ε_{rs} and ρ_{rs} in eqns (33) and (54) may be calculated directly in terms of the principal invariants of $\mathbf{C}$.

In reality, from eqn (89) we know that the eigenvalues of $\mathbf{U}$ are the positive roots of the sextic

$$\lambda^6 - \mathrm{I}_C\lambda^4 + \mathrm{II}_C\lambda^2 - \mathrm{III}_C = 0, \tag{90}$$

where the coefficients are the principal invariants of $\mathbf{C}$, given by

$$\mathrm{I}_C = \mathrm{tr}\,\mathbf{C}, \quad \mathrm{II}_C = ((\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}\,\mathbf{C}^2)/2, \quad \mathrm{III}_C = \det\mathbf{C}. \tag{91}$$

Thus, the three eigenvalues (possibly repeated) of $\mathbf{U}$ are given by

$$\lambda_i = \frac{1}{\sqrt{3}}\left(\mathrm{I}_C + 2\sqrt{\mathrm{I}_C^2 - 3\mathrm{II}_C}\cos\left(\tfrac{1}{3}(\phi - 2\pi i)\right)\right)^{1/2}, \quad i = 1, 2, 3, \tag{92}$$

where

$$\phi = \cos^{-1}\left(\frac{2\mathrm{I}_C^3 - 9\mathrm{I}_C\mathrm{II}_C + 27\mathrm{III}_C}{2(\mathrm{I}_C^2 - 3\mathrm{II}_C)^{3/2}}\right). \tag{93}$$

From the above all the distinct eigenvalues of $\mathbf{U}$, $\lambda_1, \ldots, \lambda_m$, are known and hence the coefficients ε_{rs} and ρ_{rs} are determined. For the latter, we mention that as the functions of the distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, the coefficients ε_{rs} and ρ_{rs} are independent of the order of $\lambda_1, \ldots, \lambda_m$, since ε_{rs} and ρ_{rs} are symmetric functions of $\lambda_1, \ldots, \lambda_m$.

Acknowledgements—The author dedicates this paper to the memory of his mentor, Professor Guo Zhong-Heng, who offered him much advice, help and encouragement, who passed away suddenly on 22 September 1993. The author is sincerely obliged to Professor C. S. Man for sending him his papers. Moreover, the author greatly appreciates the referee's valuable comments, which drew his attention to determination of the coefficients ε_{rs} and ρ_{rs} in terms of the principal invariants of $\mathbf{C}$. This research was supported by National Natural Science Foundation of China and the National Postdoctoral Science Foundation of China.

REFERENCES

Carlson, D. E. and Hoger, A. (1986). The derivative of a tensor-valued function of a tensor. *Q. Appl. Math.* **44**, 409–423.

Guo, Z. H. (1984). Rates of stretch tensors. *J. Elasticity* **14**, 263–267.

Guo, Z. H. and Dubey, R. N. (1984). Basic aspects of Hill's method in solid mechanics. *SM Arch.* **9**, 353–380.

Guo, Z. H., Lehmann, Th. and Liang, H. Y. (1991). Further remarks on rates of stretch tensors. *Trans. CSME* **15**, 161–172.

Guo, Z. H., Lehmann, Th., Liang, H. Y. and Man, C. S. (1992). Twirl tensor and the tensor equation $\mathbf{AX}-\mathbf{XA} = \mathbf{C}$. *J. Elasticity* **27**, 227–242.

Guo, Z. H., Li, J. B., Xiao, H. and Chen, Y. M. (1994). Intrinsic solution to the n-dimensional tensor equation $\Sigma_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}$. *Comput. Meth. Appl. Mech. Engng* **115**, 359–364.

Guo, Z. H. and Man, C. S. (1992). Conjugate stress and tensor equation $\Sigma_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}$. *Int. J. Solids Structures* **29**, 2063–2076.

Guo, Z. H., Xiao, H., Chen, Y. M. and Li, J. B. (1995). Intrinsic solution to the n-dimensional tensor equation $\mathbf{AX}-\mathbf{XA} = \mathbf{C}$. *Scient. Sinica A* (to appear).

Gurtin, M. E. and Spear, K. (1983). On the relationship between the logarithmic strain rate and the stretching tensor. *Int. J. Solids Structures* **19**, 437–444.

Hill, R. (1968). On constitutive inequalities for simple materials—I. *J. Mech. Phys. Solids* **16**, 229–242.

Hill, R. (1978). Aspects of invariance in solid mechanics. In *Advances in Applied Mechanics* (Edited by C. S. Yih), Vol. 18, pp. 1–75. Academic Press, New York.

Hoger, A. (1986). The material time derivative of logarithmic strain. *Int. J. Solids Structures* **22**, 1019–1032.

Hoger, A. (1987). The stress conjugate to logarithmic strain. *Int. J. Solids Structures* **23**, 1645–1656.

Hoger, A. and Carlson, D. E. (1984a). Determination of the stretch and rotation in the polar decomposition of the deformation gradient. *Q. Appl. Math.* **42**, 113–117.

Hoger, A. and Carlson, D. E. (1984b). On the derivative of the square root of a tensor and Guo's rate theorem. *J. Elasticity* **14**, 329–336.

Lehmann, Th. and Liang, H. Y. (1993). The stress conjugate to the logarithmic strain ln $\mathbf{V}$. *Z. Angew. Math. Mech.* **73**, 357–363.

Luehr, C. P. and Rubin, M. B. (1990). The significance of projector operators in the spectral representation of symmetric second order tensors. *Comput. Meth. Appl. Mech. Engng* **84**, 243–246.

Macvean, D. B. (1968). Die Elementararbeit in einem Kontinuum und die Zuordnung von Spannungs- und Verzerrungstensoren. *Z. Angew. Math. Phys.* **19**, 157–185.

Man, C. S. and Guo, Z. H. (1993). A basis-free formula for time rate of Hill's strain tensors. *Int. J. Solids Structures* **30**, 2819–2842.

Marsden, J. E. and Hughes, T. (1983). *Mathematical Foundation of Elasticity*. Prentice-Hall, Englewood Cliffs, NJ.

Mehrabadi, M. M. and Nemat-Nassert, S. (1987). Some basic kinematical relations for finite deformations of continua. *Mech. Mater.* **6**, 127–138.

Sawyers, K. (1986). Comments on the paper determination of the stretch and rotation in the polar decomposition of the deformation gradient. *Q. Appl. Math.* **44**, 309–311.

Scheidler, M. (1991a). Time rates of generalized strain tensors. Part I: component formulas. *Mech. Mater.* **11**, 199–210.

Scheidler, M. (1991b). Time rates of generalized strain tensors. Part II: approximate basis-free formulas. *Mech. Mater.* **11**, 211–219.

Scheidler, M. (1992). Time rates of generalized strain tensors with applications to elasticity. In *Proc. 12th Army Symp. on Solid Mechanics*, 4–7 November 1991, Plymouth, MA (Edited by S. C. Chou), pp. 59–71.

Seth, B. R. (1964). Generalized strain measures with applications to physical problems. In *Second-order Effects in Elasticity, Plasticity and Fluid Dynamics* (Edited by M. Reiner and D. Abir), pp. 162–172. Pergamon Press, Oxford.

Ting, T. C. T. (1985). Determination of $\mathbf{C}^{1/2}$, $\mathbf{C}^{-1/2}$ and more general isotropic tensor functions of $\mathbf{C}$. *J. Elasticity* **15**, 319–323.

Wang, W. B. and Duan, Z. P. (1991). On the invariant representation of spin tensors with applications. *Int. J. Solids Structures* **27**, 329–341.